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SHORTER COMMUNICATIONS

RADIATIVE HEAT TRANSFER BETWEEN PARALLEL PLATES AND CONCENTRIC CYLINDERS

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NOMENCLATURE

I,	radiation intensity;
x,	nondimensional co-ordinate in plane geometry;
μ,	cosine of angular variable in plane case;
d,	optical distance between plates;
η,	Heaviside function;
u,	total energy density;
E_n,	exponential functions;
J,	variational functional;
q, q_r	normalized heat transfer;
γ,	Euler's constant;
r, r',	nondimensional radial co-ordinate in cylindrical geometry
R_1, R_2,	radii of inner and outer cylinders respectively;
Ω,	unit angular vector;
θ, ϕ,	angular variable in cylindrical case;
$\mathbf{n}_r$,	unit vector in the radial direction;
p, L, H, C_{11}, $(C_{11})_1$, $(C_{11})_2$, C_1,	auxilliary functions defined in the text;
ρ, z, k, t, s,	Auxilliary variables;
I_n, K_n,	modified Bessel functions.

1. INTRODUCTION

THE PROBLEMS of steady state radiative heat transfer between infinite parallel plates and concentric cylinders have been studied extensively in the literature (e.g., see Refs [1–3] for a detailed bibliography). For the former problem several numerical and theoretical (invariant imbedding, Case's method) methods have been used while the latter problem has been solved only by the use of the Monte Carlo technique.

Though the variational methods have been used quite extensively to study various problems in neutron transport (e.g., see Refs [4, 5]) and rarefied gas dynamics (e.g., see Refs [6, 7]) their use in radiative transfer [8] has been still rather limited. In this note we show that for the above problems the variational methods yield very accurate results with relatively less computational effort. For the simplicity we shall treat the case of a grey gas in radiative and local thermodynamic equilibrium (see Refs [1, 2]) enclosed between diffusely reflecting surfaces though it appears that the method can be easily applied to more complicated cases. We note that for this problem it is adequate [2, 3] to consider the fundamental case in which both the walls are black and one of them is at zero absolute temperature. In this note only the method and the results will be stressed, since the details regarding the formulation of the problem etc. can be found elsewhere (e.g., see Refs [1, 2]).

2. PLANE PARALLEL PLATES

Here we consider the equation (2)

$$\mu \frac{\partial I}{\partial x} + I(x, \mu) = \frac{1}{2} \int_{-1}^{1} I(x, \mu')\, d\mu' \tag{1}$$

with the boundary conditions:

$$I\left(-\frac{d}{2}, \mu\right) = 1, \quad \mu > 0$$
$$I\left(\frac{d}{2}, \mu\right) = 0, \quad \mu < 0 \tag{2}$$

integrating equation (1) we get,

$$I(x, \mu) = \eta(\mu) \exp\left[-\left(x + \frac{d}{2}\right)\Big/\mu\right] + \frac{1}{2\mu}\left\{\eta(\mu) \int_{-d/2}^{x} dx'\, u(x') \exp\left[-(x - x')/\mu\right] - \eta(-\mu) \int_{x}^{d/2} dx'\, u(x') \exp\left[-(x - x')/\mu\right]\right\} \tag{3}$$

in which

$$\eta(\mu) = 1, \quad \mu > 0; \quad \eta(\mu) = 0, \quad \mu < 0$$

and

$$u(x) = \tfrac{1}{2} \int_{-1}^{1} I(x, \mu)\,\mathrm{d}\mu \qquad (4)$$

For $u(x)$, equation (3) is easily converted to an integral equation and we have

$$u(x) = \left(E_2\ \ x + \frac{d}{2}\right) + \frac{1}{2}\int_{-d/2}^{d/2} \mathrm{d}x'\, E_1(|x - x'|)\, u(x') \qquad (5)$$

in which $E_n(x)$ are the well known exponential functions [10]. Equation (5) can be derived from the functional [4–7]

$$J(\tilde{u}) = \int_{-d/2}^{d/2} \mathrm{d}x\, \tilde{u}(x)\left[\tilde{u}(x) - \frac{1}{2}\int_{-d/2}^{d/2} \mathrm{d}x'\, E_1(|x - x'|)\, \tilde{u}(x') - 2E_2\left(x + \frac{d}{2}\right)\right] \qquad (6)$$

which takes a minimum value for $\tilde{u} = u$, the solution of equation (5), and

$$J_{\min}(\tilde{u}) = -\int_{-d/2}^{-d} \mathrm{d}x\, u(x)\, E_2\left(x + \frac{d}{2}\right) \qquad (7)$$

Now the normalized heat transfer between the plates is given by

$$q(x) = 2\int_{-1}^{1} \mathrm{d}\mu\, \mu\, I(x, \mu) \qquad (8)$$

and is easily shown to be constant. Now evaluating this at $x = -d/2$ and using equation (3), after some algebra we find that

$$q = 1 + J_{\min}(\tilde{u}) \qquad (9)$$

for $\tilde{u}$ we considered a simple trial function $Ax + B$. The numerical results for different values of d are given in Table 1, and are in excellent agreement with the "exact" results [9]. In this, the following integrals were useful:

$$\int_{-d/2}^{d/2} x^n E_m\left(x + \frac{d}{2}\right)\mathrm{d}x = \sum_{r=0}^{n}\left(\frac{d}{2}\right)^{n-r} \times \frac{n!}{(n-r)!}\left\{\frac{(-1)^{n-r}}{m+r} - E_{m+r+1}(d)\right\}$$

$$\int_{-d/2}^{d/2} x'^n E_1(|x - x'|)\,\mathrm{d}x' = \sum_{r=0}^{n}\frac{n!}{(n-r)!} \times \left\{\frac{x^{n-r}}{r+1}[(-1)^r + 1] - \left(\frac{d}{2}\right)^{n-r} \times \left[(-1)^r E_{r+2}\left(\frac{d}{2} + x\right) + E_{r+2}\left(\frac{d}{2} - x\right)\right]\right\}$$

Further for the above trial function, using the asymptotic expansions for E_n function (10), it is easily shown that

For $d \ll 1$,

$$q = 1 - d + \frac{d^2}{2}[\psi(3) - \log d] + \frac{d^3}{3}\{\tfrac{1}{2} + \tfrac{1}{2}\log d + [\psi(4) - \tfrac{3}{2}\psi(3)]^2\} \qquad (10)$$

and for $d \gg 1$,

$$q = \frac{4}{3}\frac{1}{d} - \frac{17}{9}\frac{1}{d^2} + \frac{8}{3}\frac{1}{d^3} - \frac{34}{9}\frac{1}{d^4} + \frac{2}{d}e^{-d} + \ldots \qquad (11a)$$

Here

$$\psi(m) = -\gamma + \sum_{n=1}^{m-1}\frac{1}{n},\ n > 1$$

Table 1. Radiative heat transfer between parallel plates

Optical thickness d	"Exact" [Ref. 9]	Variational	$d \ll 1$ Equation (10)	$d \gg 1$ Equation (11a)	$d \gg 1$ Equation (11b)
0·1	0·9157	0·9157	0·9159		
0·2	0·8491	0·8492	0·8498		
0·5	0·7040	0·7042	0·7090		
1·0	0·5532	0·5534			
2·0	0·3900	0·3901			
3·0	0·3016	0·3017		0·3199	0·3016
4·0		0·2461		0·2513	0·2459
5·0		0·2077		0·2091	0·2076
6·0		0·1797		0·1800	0·1797
7·0		0·1584		0·1584	0·1583
8·0		0·1416		0·1416	0·1415
9·0		0·1280		0·1280	0·1279
10·0		0·1168		0·1167	0·1167

$$\psi(1) = -\gamma$$
$$\gamma = 0{\cdot}57721\ldots.$$

We note that for $d \gg 1$, Ferziger and Simmons [2] and Heaslet and Warming [11] also give a simple and very useful formula:

$$q = \frac{\frac{4}{3}}{d + 2z_0} \tag{11b}$$

in which,

$$z_0 = 0{\cdot}710446.$$

The numerical values corresponding to the expression (10), (11a) and (11b) are also given in Table 1.

3. CONCENTRIC CYLINDERS

For two concentric cylinders of radii R_1 and R_2, using cylindrical symmetry, we easily find that

$$I(r, \theta, \phi) = \eta(\phi_0 - \phi)\exp(-|\mathbf{r} - \mathbf{R}_1|) + \{\eta(\phi_0 - \phi)\int_0^{|\mathbf{r}-\mathbf{R}_1|} \mathrm{d}|\mathbf{r} - \mathbf{r}'|\exp(-|\mathbf{r} - \mathbf{r}'|)u(r') + \eta(\phi - \phi_0)\int_0^{|\mathbf{r}-\mathbf{R}_2|} \mathrm{d}|\mathbf{r} - \mathbf{r}'|\exp(-|\mathbf{r} - \mathbf{r}'|)u(r') . \tag{12}$$

Here r is the radial distance of a point from the axis of the cylinders, and $\Omega = (\theta, \phi)$ is the direction of the line of observation. Also,

$$u(r) = \frac{1}{4\pi}\int I(r, \theta, \phi)\,\mathrm{d}\Omega \tag{13}$$

$$|\mathbf{r} - \mathbf{r}'| = \frac{r\cos\phi \pm (r'^2 - r^2\sin^2\phi)^{\frac{1}{2}}}{\sin\theta} \tag{14}$$

and

$$\phi_0 = \sin^{-1}\frac{R_1}{r}. \tag{15}$$

Now for $u(r)$, equation (12) is easily converted into an integral equation:

$$u(r) = p(r) + Lu(r) \tag{16}$$

in which:

$$p(r) = \frac{1}{4\pi}\int_{-1}^{1}\mathrm{d}\mu\int_{-\phi_0}^{\phi_0}\mathrm{d}\phi\exp(-|\mathbf{r} - \mathbf{R}_1|),\ \mu = \cos\theta \tag{17}$$

and

$$Lu(r) = \frac{1}{4\pi}\left\{\int_{-1}^{1}\mathrm{d}\mu\int_{-\phi_0}^{\phi_0}\mathrm{d}\phi\int_0^{|\mathbf{r}-\mathbf{R}_2|}\mathrm{d}|\mathbf{r} - \mathbf{r}'|\exp(-|\mathbf{r} - \mathbf{r}'|)u(r') + \int_{-1}^{1}\mathrm{d}\mu\int_{\phi_0}^{2\pi-\phi_0}\mathrm{d}\phi\int_0^{|\mathbf{r}-\mathbf{R}_2|}\mathrm{d}|\mathbf{r} - \mathbf{r}'|\exp(-|\mathbf{r} - \mathbf{r}'|)u(r')\right\}. \tag{18}$$

Now the normalized radial heat transfer between the cylinders is given by

$$q_r = \frac{1}{\pi}\int \Omega\,.\,\mathbf{n}_r\,I(r, \Omega)\,\mathrm{d}\Omega \tag{19}$$

where $\mathbf{n}_r$ is a unit vector in the radial direction. Therefore,

$$q_r = \frac{1}{\pi}\int_0^{\pi}\mathrm{d}\theta\sin\theta\cos\theta\int_0^{2\pi}\mathrm{d}\phi\cos\phi\,I(r, \theta, \phi) \tag{20}$$

and is easily shown to be a constant. Evaluating this at $r = R_1$, after a few transformations and some algebra we find that

$$q_r = 1 - \frac{4}{\pi R_1}\int_{R_1}^{R_2}\mathrm{d}r\,r\,u(r)\int_{(r-R_1)}^{(r^2-R_1^2)^{\frac{1}{2}}}\mathrm{d}\rho \times \frac{\rho[r^2 - (R_1^2 + \rho^2)]}{\{[(R_1 + r)^2 - \rho^2][\rho^2 - (R_1 - r)^2]\}^{\frac{1}{2}}} \times \int_0^{\infty}\mathrm{d}z\frac{\exp[-(z^2 + \rho^2)^{\frac{1}{2}}]}{(z^2 + \rho^2)^{\frac{1}{2}}}. \tag{21}$$

Also after some transformations we find that

$$p(r) = \frac{1}{\pi}\int_{r-R_1}^{(r^2-R_1^2)^{\frac{1}{2}}}\mathrm{d}\rho\frac{\rho\{r^2 - (R_1^2 + \rho^2)\}}{\{[(\rho + r)^2 - R_1^2][R_1^2 - (\rho - r)^2]\}^{\frac{1}{2}}} \times \int_0^{\infty}\mathrm{d}z\frac{\exp[-(\rho^2 + z^2)^{\frac{1}{2}}]}{(\rho^2 + z^2)^{\frac{1}{2}}}. \tag{22}$$

Thus

$$q_r = 1 - \frac{4}{R_1}\int_{R_1}^{R_2}\mathrm{d}r\,r\,u(r)\,p(r). \tag{23}$$

Now equation (16) is equivalent to minimizing the functional

$$J(\tilde{u}) = \int_{R_1}^{R_2}\mathrm{d}r\,r\,\tilde{u}(r)[\tilde{u}(r) - L\tilde{u}(r) - 2p(r)] \tag{24}$$

and

$$J_{\min}(\tilde{u}) = -\int_{R_1}^{R_2}\mathrm{d}r\,r\,u(r)\,p(r). \tag{25}$$

Hence $J_{\min}$ is directly related to q_r, and we have

$$q_r = 1 + \frac{4}{R_1}J_{\min}(\tilde{u}). \tag{26}$$

Now for the trial function

$$\tilde{u}(r) = A\log r/R_2 \tag{27}$$

we find that

$$J(\tilde{u}) = A^2C_{11} + AC_1. \tag{28}$$

The expressions for C_{11} and C_1, which at first appear rather complicated, can be considerably simplified by use of several transformations and some lengthy algebra. Thus, we find that

$$C_1 = \frac{2R_1}{\pi} \int_{\pi/2}^{\pi} d\phi \cos\phi \int_0^{R_1\cos\phi + (R_2^2 - R_1^2\sin^2\phi)^{\frac{1}{2}}} d\rho \times \log\left[\{R_1^2 + \rho^2 - 2R_1\rho\cos\phi\}^{\frac{1}{2}}/R_2\right] Ki_2(\rho) \quad (29)$$

in which $Ki_n(x)$, sometimes known as Bickley functions, are the well studied [10] integrals of modified Bessel functions, i.e.,

$$Ki_n(x) = \int_x^\infty K_{n-1}(t)\,dt \quad (30)$$

in particular

$$Ki_2(x) = x\{K_1(x) - Ki_1(x)\} \quad (31)$$

and

$$C_{11} = (C_{11})_1 + (C_{11})_2 \quad (32)$$

in which

$$\begin{aligned}(C_{11})_1 = {} & \left(R_1 \log\frac{R_1}{R_2}\right)^2 \int_0^1 dk\, I_1\left(\frac{R_1}{k}\right) K_1\left(\frac{R_1}{k}\right) \\ & - 2R_1 \log\frac{R_1}{R_2} \int_0^1 dk\, k\, I_1\left(\frac{R_1}{R}\right) K_0\left(\frac{R_2}{k}\right) \\ & + R_1 \log\frac{R_1}{R_2} \int_0^1 dk\, k \left\{ I_1\left(\frac{R_1}{k}\right) K_0\left(\frac{R_1}{k}\right) - I_0\left(\frac{R_1}{k}\right) K_1\left(\frac{R_1}{k}\right)\right\} \\ & + 2\int_0^1 dk\, k^2 I_0\left(\frac{R_1}{k}\right) K_0\left(\frac{R_2}{k}\right) \\ & - \int_0^1 dk\, k^2 I_0\left(\frac{R_1}{k}\right) K_0\left(\frac{R_1}{k}\right) \\ & - \int_0^1 dk\, k^2 I_0\left(\frac{R_2}{k}\right) K_0\left(\frac{R_2}{k}\right) \end{aligned} \quad (33)$$

and,

$$(C_{11})_2 = \frac{R_1}{\pi} \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\phi \cos\phi \exp(-2R_1\cos\phi\sec\theta) \times H^2(\theta, \phi, R_1, R_2) \quad (34)$$

in which

$$H(\theta, \phi, R_1, R_2) = \int_0^{R_1\cos\phi + (R_2^2 - R_1^2\sin^2\phi)^{\frac{1}{2}}} ds \exp(-s\sec\phi) \times \log\left[\frac{R_1}{R_2}\left\{\left(\frac{s}{R_1}\right)^2 + 2\left(\frac{s}{R_1}\right)\cos\phi + 1\right\}^{\frac{1}{2}}\right]. \quad (35)$$

Here I and K are modified Bessel functions. The integrals in above expressions can be evaluated quite conveniently by use of Gaussian quadratures (see used a 10-point formula for a single integral). The numerical results for a few selected

Table 2. Radiative heat transfer between concentric cylinders

Optical thickness $R_2 - R_1$	Radius ratio R_1/R_2		
	0·1	0·5	0·9
0·1	0·9893	0·9677	0·9462
0·5	0·9464	0·8476	0·7688
1·0	0·8937	0·7225	0·6167
2·0	0·7956	0·5446	0·4371
3·0	0·7105	0·4313	0·3367
4·0	0·6377	0·3549	0·2727
5·0	0·5763	0·3010	0·2291
6·0	0·5250	0·2615	0·1976
7·0	0·4810	0·2308	0·1738
8·0	0·4429	0·2060	0·1549
9·0	0·4102	0·1864	0·1398
10·0	0·3821	0·1703	0·1278

Table 3. Radiative heat transfer between concentric cylinders (Heaslet and Baldwin, Ref. [12])

$$q \sim \frac{1}{1 + \frac{3R_1}{4}\log\frac{R_2}{R_1}}$$

Optical thickness $R_2 - R_1$	Radius ratio R_1/R_2		
	0·1	0·5	0·9
0·1	0·9811	0·9505	0·9336
0·5	0·9214	0·7936	0·7376
1·0	0·8390	0·6579	0·5843
2·0	0·7226	0·4902	0·4128
3·0	0·6346	0·3906	0·3191
4·0	0·5657	0·3247	0·2600
5·0	0·5103	0·2778	0·2949
6·0	0·4648	0·2427	0·1898
7·0	0·4267	0·2155	0·1672
8·0	0·3944	0·1938	0·1494
9·0	0·3667	0·1760	0·1351
10·0	0·3426	0·1613	0·1232

values of radius ratio R_1/R_2 and the optical thickness $R_2 - R_1$ are given in Table 2. These results appear in excellent agreement with Monte Carlo results of Perlmutter and Howell [3]. In fact in view of our results for parallel plates, we can assert that results reported in Table 2 are more accurate than results of Ref. [3].

We note that for this case, Heaslet and Baldwin [12] give an approximate relation,

$$q \sim \frac{1}{1 + \frac{3}{4}R_1 \log R_2/R_1} \qquad (36)$$

The results corresponding to this expression are given in Table 3, and appear to compare somewhat less favorably with the results given in Table 2.

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OVERALL CONSTRICTION RESISTANCE BETWEEN CONTACTING ROUGH, WAVY SURFACES†

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NOMENCLATURE

- a, contact spot radius;
- A, contour area radius;
- b, heat flux tube radius;
- B, heat channel radius;
- k, thermal conductivity, $k = 2k_1k_2/(k_1 + k_2)$;
- N, number of contact spots;
- R, overall thermal contact resistance, [°C/W].

Greek characters
- β, maldistribution factor $(1 < \beta < 1{\cdot}4)$;
- γ, ratio A/B;
- ϵ, ratio a/b;
- ψ, constriction factor defined by equations (2) and (3).

Subscripts
- 1, 2, metals 1 and 2;
- o, microscopic;
- c, macroscopic;

Superscript
- T, factor based upon uniform temperature.

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INTRODUCTION

In a recent article [1] the authors showed qualitatively